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Note

# A generalization of Naundorf's fixpoint theorem

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**Abstract**

Given is an ordered set in which every chain has an upper bound and every pair of elements has a greatest lower bound. Let $Z$ be its set of maximal elements and let $F$ be a function from $Z$ to $Z$. A condition is presented that implies that $F$ has a unique fixpoint. This is a generalization of a theorem of Naundorf. In Naundorf's theorem, the condition is related to causality for behaviour that develops in time. 

*Keywords:* Fixpoint theorem; Semilattice; Zorns lemma; Strict causality

## 1. Introduction

In fundamental computer science, fixpoint equations play the same role as differential equations in mathematical physics. They summarize the results of detailed modelling and are the starting points for formal analysis. Fixpoint equations are often disguised as inductive definitions. This is more appealing to the intuition, but it may hamper the analysis and, occasionally, even lead to unsound reasoning. It is therefore important to recognize fixpoint equations and to investigate their solutions.

This note is written to clarify and generalize a new fixpoint theorem due to Naundorf, cf. [4]. We use Feijen's linear proof format, cf. [2], to show at which points which conditions are used.

Naundorf's theorem is stated in the following context. Let $(T, \leqslant)$ be a partially ordered set, interpreted as the time domain, and let $V$ be a nonempty set. The set $Z = (T \to V)$ of functions from $T$ to $V$ is used to model behaviour evaluating in time.

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For $u,\ v \in Z$, the set $Head(u, v)$ is defined as the set of elements $t \in T$ such that $u.s = v.s$ for all $s \leqslant t$. Note that we use the infix dot for function application.

A function $F \in Z \to Z$ is called *strictly causal* iff, for every pair $u,\ v \in Z$ with $u \neq v$, the set $Head(u, v)$ is a strict subset of $Head(F.u, F.v)$. Naundorf's theorem then reads

**Theorem A.** Let $F \in Z \to Z$ be strictly causal.
(a) Then function $F$ has a unique fixpoint, say $z_0$, in $Z$.
(b) Every $z \in Z$ satisfies $Head(z, F.z) = Head(z, z_0)$.

Naundorf's proof is based on the set $X$ of the partial functions $x \in T \rightharpoonup V$ for which the domain $dom.x$ is an initial subset of $T$ (i.e., if $t \in dom.x$ then $s \in dom.x$ for all $s \leqslant t$). The elements of $X$ are regarded as relations (subsets of $T \times V$) and ordered by inclusion. Then $X$ is a cpo, in the sense that every chain (linearly ordered subset) has a least upper bound in $X$. Moreover, the set $Z$ is precisely the set of maximal elements of $X$.

Naundorf begins his proof with the observation that strict causality of $F$ implies that, for all $u,\ v \in Z$ and all $x \in X$,

$$
\begin{aligned}
& x \sqsubset u \wedge x \sqsubset v \wedge x \sqsubset F.u \\
& \quad \Rightarrow (\exists y \in X \ \cdot \ x \sqsubset y \wedge y \sqsubseteq F.u \wedge y \sqsubseteq F.v).
\end{aligned} \tag{0}
$$

Here, $\sqsubset$ is the irreflexive conjunction of $\sqsubseteq$ and $\neq$.

The set $X$ is not closed under finite intersections in the set of relations. Yet, $X$ is a semilattice: every pair $x,\ y \in X$ has a greatest lower bound $x \sqcap y$ in $X$, which is the partial function $x|Head(x, y)$ where $Head(x, y)$ is the set of elements $t$ in the intersection of the domains of $x$ and $y$ such that $x.s = y.s$ for all $s \leqslant t$.

If we use these greatest lower bounds, condition (0) implies the simpler condition that, for all $u,\ v \in Z$,

$$
u \neq v \Rightarrow u \sqcap v \sqcap F.u \sqsubset F.u \sqcap F.v. \tag{1}
$$

This is shown as follows. If $u \neq v$, then $x = u \sqcap v \sqcap F.u$ is not maximal in $X$; this implies $x \notin Z$; therefore $x$ differs from $u$, $v$, and $F.u$; then (0) implies that $x \sqsubset F.u \sqcap F.v$.

## 2. The generalization

We now take a more abstract point of view. A partially ordered set $(X, \sqsubseteq)$ is called *inductive* iff every chain (linearly ordered subset) in $X$ has an upper bound in $X$, cf. [1]. Note that this is a weaker condition than being a cpo, since every chain in a cpo must have a *least* upper bound. According to [1], Zorn's Lemma asserts that every inductive poset has a maximal element. We need the following stronger version:

**Lemma.** Let $L$ be a chain in an inductive poset $X$. Then $L$ has an upper bound which is a maximal element of $X$.

**Proof.** Let $W$ be the set of upper bounds of $L$ in $X$. For every chain $C$ in $W$, the set $L \cup C$ is a chain in $X$, which has an upper bound in $X$; this upper bound then is an element of $W$ and an upper bound of $C$. This shows that the poset $W$ is inductive. By Zorn's Lemma, it follows that $W$ has a maximal element, say $w$. If $x \in X$ satisfies $w \sqsubseteq x$, then $x \in W$ and hence $w = x$ by maximality of $w$. So, $w$ is a maximal element of $X$ and also an upper bound of $L$. □

As above, we use the definition that a poset $(X, \sqsubseteq)$ is a *semilattice* iff every pair of elements $x$, $y$ has a greatest lower bound $x \sqcap y$. Since strict causality implies (0), and (0) implies (1), it is easy to see that Theorem A above follows from

**Theorem B.** Let $(X, \sqsubseteq)$ be an inductive semilattice. Let $Z$ be the set of maximal elements of $X$ and let $F \in Z \to Z$ be such that (1) holds for every pair of elements $u$, $v$ in $Z$.
(a) Function $F$ has a unique fixpoint, say $z_0$, in $Z$.
(b) Every $z \in Z$ satisfies $z \sqcap F.z = z \sqcap z_0$.

Condition (1), though nicer than (0), is still not an appealing requirement. We therefore, weaken it further. We claim that it implies the two conditions

$$u \sqcap F.u \sqsubseteq v \Rightarrow u \sqcap F.u \sqsubseteq F.v, \tag{2}$$

$$F.u \sqcap F.v = u \sqcap v \Rightarrow u = v. \tag{3}$$

Indeed, condition (2) holds trivially for $u = v$ and follows from (1) for $u \neq v$. Condition (3) also follows from (1), since $F.u \sqcap F.v = u \sqcap v$ implies $u \sqcap v \sqcap F.u = F.u \sqcap F.v$ and, hence, $u = v$ by contraposition of (1).

We now generalize Theorem B by claiming

**Theorem C.** Let $(X, \sqsubseteq)$ be an inductive semilattice. Let $Z$ be the set of maximal elements of $X$ and let $F \in Z \to Z$ be such that (2) and (3) hold for every pair of elements $u$, $v$ in $Z$.
(a) Function $F$ has a unique fixpoint, say $z_0$, in $Z$.
(b) Every $z \in Z$ satisfies $z \sqcap F.z \sqsubseteq z_0$.

Before proving Theorem C, we first show that it implies Theorem B. It suffices to treat part (b) of Theorem B.

**Proof of Theorem B.** Part (b) follows from the observation that, for any $z \in Z$, we have

$$z \sqcap F.z = z \sqcap z_0$$
$$\equiv \{\text{calculus}\}$$
$$z \sqcap F.z \sqsubseteq z_0 \wedge z \sqcap z_0 \sqsubseteq F.z$$

$$\Leftarrow \quad \{\text{Theorem C (b) and } F.z_0 = z_0\}$$
$$z_0 \sqcap z \sqcap F.z_0 \sqsubseteq F.z_0 \sqcap F.z$$
$$\equiv \quad \{\text{for } z \neq z_0 \text{ use (1) with } u := z_0 \text{ and } v := z\}$$
$$\text{true.}$$

This proves Theorem B from Theorem C. □

We turn to the proof of Theorem C.

**Proof of Theorem C.** It is clear that, if $u$ and $v$ are fixpoints of $F$, condition (3) implies $u = v$. This proves that function $F$ has at most one fixpoint. The main task therefore, is to construct a fixpoint.

Inspired by condition (2), we define the subset $M$ of $X$ by

$$x \in M \equiv (\forall z \in Z \cdot x \sqsubseteq z \Rightarrow x \sqsubseteq F.z).$$

The definition of $M$ together with condition (2) immediately implies that, for any $u \in Z$,

$$u \sqcap F.u \in M. \tag{4}$$

We now claim that the poset $M$ is inductive. This is proved as follows. Let $L$ be a chain in $M$. We have to prove that $L$ has an upper bound in $M$. Since $X$ is inductive, the Lemma implies that $L$ has an upper bound $z \in Z$. Since $L \subseteq M$, the definition of $M$ yields that $F.z$ is also an upper bound of $L$. Therefore, $z \sqcap F.z$ is an upper bound of $L$, which is contained in $M$ because of Formula (4).

Since $M$ is inductive, Zorn's Lemma implies that $M$ has at least one maximal element. For every maximal element $y$ of $M$ and every element $z \in Z$, we now observe

$$y \sqsubseteq z$$
$$\Rightarrow \quad \{y \in M \text{ and definitions of } M \text{ and } \sqcap\}$$
$$y \sqsubseteq z \sqcap F.z$$
$$\Rightarrow \quad \{y \text{ is maximal in } M \text{ and } z \sqcap F.z \in M \text{ by (4)}\}$$
$$y = z \sqcap F.z. \tag{5}$$

For every maximal element $y$ of $M$ and every element $z \in Z$, we subsequently observe

$$y \sqsubseteq z$$
$$\Rightarrow \quad \{(5) \text{ twice, once with } z := F.z\}$$
$$y = z \sqcap F.z \wedge y = F.z \sqcap F.(F.z)$$
$$\Rightarrow \quad \{(3) \text{ with } u := z \text{ and } v := F.z\}$$
$$y = z \sqcap F.z \wedge z = F.z$$
$$\Rightarrow \quad \{\sqcap \text{ is idempotent, } z \in Z\}$$
$$y \in Z \wedge y = F.y. \tag{6}$$

Since $X$ is inductive, the Lemma implies that every element of $X$ has an upper bound in $Z$. Therefore, calculation (6) implies that every maximal element of $M$ is an element of $Z$ and a fixpoint of $F$. Since $M$ has maximal elements and since $F$ has at most one fixpoint, this proves that $F$ has a unique fixpoint, which is the unique maximal element of $M$.

Since $M$ is inductive, the Lemma implies that every element of $M$ has an upper bound equal to the maximal element of $M$, i.e., to the fixpoint of $F$. Therefore, property (b) of Theorem C follows from (4). This concludes the proof of Theorem C. □

**Remark.** In Theorem C, the assumption that $X$ is inductive cannot be weakened to the assumption that every element of $X$ has an upper bound in $Z$. This is shown as follows. Let $X$ be the set $\mathbb{N}$ of the natural numbers including 0. Let $Z$ be the set of the odd natural numbers. Let $\sqsubseteq$ be defined by

$$x \sqsubseteq y \equiv x \leqslant y \wedge (x \notin Z \vee x = y).$$

Then $(X, \sqsubseteq)$ is a semilattice, $Z$ is its set of maximal elements and every element of $X$ has an upper bound in $Z$. The function $F \in Z \to Z$ given by $F.z = z + 2$ satisfies (2) and (3) but has no fixpoint.

It may also be useful to note that condition (2) is not sufficient to imply existence of a fixpoint. This is shown as follows. Let $Z$ be an arbitrary set. Let $X$ be the set of the nonempty finite subsets of $Z$ ordered by containment, i.e., $x \sqsubseteq y$ iff $x \supseteq y$. Then the maximal elements of $X$ are the singleton sets, which can be identified with elements of $Z$. The operation $\sqcap$ on $X$ corresponds to the union. In this way, it is easy to see that $X$ is an inductive semilattice. Consider a function $F \in Z \to Z$. In this case, condition (2) holds if and only if $v \in \{u, F.u\}$ implies $F.v \in \{u, F.u\}$ for all $u$ and $v \in Z$. If we now take $F$ to satisfy $F.(F.z) = z$ for all $z$, we see that $F$ need not have any fixpoints. □

## 3. Concluding remarks

Theorem C can be compared with Jonker's generalization [3] of the theorem of Knaster–Tarski, cf. [5], a simplified version of which reads as follows. Let $(X, \sqsubseteq)$ be a cpo and let $f \in X \to X$ be pseudo-monotonic, i.e. such that $x \sqsubseteq y$ and $f.x \sqsubseteq y$ implies $f.x \sqsubseteq f.y$. Then $f$ has a fixpoint in $X$. Since our function $F$ is only an endofunction of the set of maximal elements, Theorem C is more or less orthogonal to this result.

It should be noted that the theorem of Knaster–Tarski cannot be generalized to our setting of inductive semilattices. For example, let $X$ be the sets of the rational numbers $x$ with $0 \leqslant x \leqslant 1$ with the usual order. Then $X$ is obviously inductive and a lattice. Yet, the function $f \in X \to X$ given by $f.x = \frac{1}{2}x^2 + \frac{1}{4}$ is monotonic, but has no fixpoint in $X$ since the roots of $x = \frac{1}{2}x^2 + \frac{1}{4}$ are irrational. This implies that our Theorem C cannot be derived from a version of the theorem of Knaster–Tarski.

In the setting of Theorem A, the condition of strict causality as proposed by Naundorf is still more natural than the conjunction of (2) and (3). In fact, when translated to the setting of the Introduction, the latter conditions are

$$Head(u, F.u) \subseteq Head(u, v) \Rightarrow Head(u, F.u) \subseteq Head(u, F.v),$$
$$Head(u, v) = Head(F.u, F.v) \subseteq Head(u, F.u) \Rightarrow u = v.$$

These conditions, though formally weaker than strict causality, seem much too complicated to be useful.

We therefore consider the main asset of the generalization to be the gain of clarity of presentation caused by the more abstract setting. A secondary point is the possibility of other applications and other generalizations of the theorem.

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